

Factorization of Difference Equations by Semiconjugacy with Application to Non-autonomous Linear Equations

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Abstract. The existence of a semiconjugate relation permits the transformation of a higher order difference equation on a group into an equivalent triangular system of two difference equations of lower orders. Introducing time-dependent form symmetries in this paper enables us to identify the semiconjugate property in a larger set of non-autonomous difference equations than previously considered. We show that there is a substantial class of equations having this feature that includes the general (non-autonomous, non-homogeneous) linear equation with variable coefficients in an arbitrary algebraic field.

1 Introduction

Difference equations of order greater than one that are of the following type

$$x_{n+1} = f_n(x_n, x_{n-1}, \dots, x_{n-k}) \quad (1)$$

determine the forward evolution of a variable x_n in discrete time since the time index or the independent variable n is integer-valued with $n \geq 0$.

In previous studies of semiconjugate factorizations of difference equations of type (1), e.g., [3], [4], [5] or [6], the form symmetry linking the higher dimensional unfolding map of the original equation to that of the lower dimensional factor was assumed to be independent of n . While this assumption did not substantially curtail the applicability of the method, it did rule out certain non-autonomous equations. For example, the method worked for non-homogeneous linear equations with constant coefficients but did not apply to linear equations with variable coefficients.

The main goal of this article is to extend the aforementioned factorization method to allow *time-dependent* form symmetries where the form symmetry may depend explicitly on the independent variable n . This extension is significant as it covers all non-autonomous equations of type (1). In particular, the extended method may be applied to general (non-autonomous, non-homogeneous) linear equations over arbitrary algebraic fields to show that such equations admit semiconjugate factorizations via eigensequences (i.e., the solutions of an associated discrete Riccati difference equation of lower order). For ease of reference we state some of the basic concepts and notation here; additional background material for this article is available in [5].

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As usual, the number k in (1) is a fixed positive integer and $k + 1$ represents the *order* of the difference equation (1). The underlying space of variables x_n is a group G and $f_n : G^{k+1} \rightarrow G$ is a given function for each $n \geq 1$. If $f_n = f$ does not explicitly depend on n then (1) is said to be *autonomous*; it is *non-autonomous* otherwise. A (forward) *solution* of Eq.(1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that is recursively generated by (1) from a set of $k + 1$ initial values $x_0, x_{-1}, \dots, x_{-k} \in G$. Forward solutions have traditionally been of greater interest in discrete models that are based on Eq.(1) although other types of solutions (e.g., those having domain \mathbb{Z} , the set of all integers) can also be readily defined.

Each f_n is “unfolded” by the associated vector map $F_n : G^{k+1} \rightarrow G^{k+1}$ that are defined as

$$F_n(u_0, \dots, u_k) = [f_n(u_0, \dots, u_k), u_0, \dots, u_{k-1}], \quad u_j \in G \text{ for } j = 0, 1, \dots, k. \quad (2)$$

The *unfoldings* F_n determine the equation

$$(y_{0,n+1}, y_{1,n+1}, \dots, y_{k,n+1}) = F_n(y_{0,n}, y_{1,n}, \dots, y_{k,n})$$

in G^{k+1} . Each vector $(y_{0,n+1}, \dots, y_{k,n+1})$ represents a *state* of the system, or of Eq.(1); G^{k+1} is the *state space*, in analogy to the phase space in differential equations.

2 Semiconjugate relation and factorization

Let F_n be the unfolding on G^{k+1} of f_n for each n . Then (1) is equivalent to

$$X_{n+1} = F_n(X_n), \quad X_n = (x_n, \dots, x_{n-k}). \quad (3)$$

We are interested in deriving a lower dimensional equation

$$Y_{n+1} = \Phi_n(Y_n), \quad Y_n = (y_n, \dots, y_{n-m+1}), \quad m \leq k \quad (4)$$

for (3). If there exists a sequence of maps $H_n : G^{k+1} \rightarrow G^m$ such that for every solution $\{X_n\}$ of (3)

$$Y_n = H_n(X_n), \quad n = 0, 1, 2, \dots \quad (5)$$

is a solution of (4) then

$$\Phi_n(H_n(X_n)) = \Phi_n(Y_n) = Y_{n+1} = H_{n+1}(X_{n+1}) = H_{n+1}(F_n(X_n)).$$

Therefore, (5) is satisfied for all solutions of (3) and (4) if and only if the sequence $\{H_n\}$ of maps satisfies the following equality for all n

$$H_{n+1} \circ F_n = \Phi_n \circ H_n. \quad (6)$$

If the mappings H_n are independent of n , i.e., $H_n = H$ for all n then Eq.(6) reduces to

$$H \circ F_n = \Phi_n \circ H \quad (7)$$

namely, the time-independent semiconjugate relation as defined in prior studies. We can now give the following more general definition.

Definition 1 Let $k \geq 1$, $1 \leq m \leq k$. If there is a sequence of surjective maps $H_n : G^{k+1} \rightarrow G^m$ such that Eq.(6) is satisfied for a given pair of function sequences $\{F_n\}$ and $\{\Phi_n\}$ then we say that F_n is **semiconjugate** to Φ_n for each n and refer to the sequence $\{H_n\}$ as a **(time-dependent) form symmetry** of Eq.(3) or equivalently, of Eq.(1). Since $m < k + 1$, the form symmetry $\{H_n\}$ is **order-reducing**.

Technically, a time-dependent form symmetry can also be defined as a *single map*

$$H : \mathbb{N} \times G^{k+1} \rightarrow G^m, \quad H(n; u_0, \dots, u_k) = H_n(u_0, \dots, u_k)$$

We choose the sequence definition due to its more intuitive content.

The following result extends its time-independent analog in [5] and makes precise the concept of semiconjugate factorization for the recursive difference equation (1).

Lemma 2 Let $k \geq 1$, $1 \leq m \leq k$, let $h_n : G^{k-m+1} \rightarrow G$ for $n \geq -m + 1$ be a sequence of functions on a given non-trivial group G and define the functions $H_n : G^{k+1} \rightarrow G^m$ by

$$H_n(u_0, \dots, u_k) = [u_0 * h_n(u_1, \dots, u_{k+1-m}), \dots, u_{m-1} * h_{n-m+1}(u_m, \dots, u_k)]. \quad (8)$$

Then the following statements are true:

- (a) The function H_n defined by (8) is surjective for each fixed $n \geq 0$.
- (b) If $\{H_n\}$ is an order-reducing form symmetry then the difference equation (1) is equivalent to the system of equations

$$t_{n+1} = \phi_n(t_n, \dots, t_{n-m+1}), \quad (9)$$

$$x_{n+1} = t_{n+1} * h_{n+1}(x_n, \dots, x_{n-k+m})^{-1} \quad (10)$$

whose orders m and $k + 1 - m$ respectively, add up to the order of (1).

- (c) The map $\Phi_n : G^m \rightarrow G^m$ in (6) is the unfolding of Eq.(9) for each $n \geq 0$; i.e., each Φ_n is of scalar type.

Proof. (a) Let n be a fixed non-negative integer and for $j = 0, \dots, m-1$ denote the j -th coordinate function of H_n by

$$\eta_{j+1}(u_0, \dots, u_k) = u_j * h_{n-j}(u_{j+1}, \dots, u_{j+k+1-m}) \quad (11)$$

Now choose an arbitrary point $(v_1, \dots, v_m) \in G^m$ and define

$$\begin{aligned} u_{m-1} &= v_m * h_{n-m+1}(u_m, u_{m+1}, \dots, u_k)^{-1}, \\ u_m &= u_{m+1} = \dots u_k = \bar{u} \end{aligned} \quad (12)$$

where \bar{u} is a fixed element of G , e.g., the identity. Then

$$\begin{aligned}
v_m &= u_{m-1} * h_{n-m+1}(\bar{u}, \bar{u} \dots, \bar{u}) \\
&= u_{m-1} * h_{n-m+1}(u_m, u_{m+1} \dots, u_k) \\
&= \eta_m(u_0, \dots, u_k) \\
&= \eta_m(u_0, \dots, u_{m-2}, \underbrace{v_m * h_{n-m+1}(\bar{u}, \bar{u} \dots, \bar{u})^{-1}}_{u_{m-1}}, \bar{u} \dots, \bar{u}).
\end{aligned}$$

for any selection of elements $u_0, \dots, u_{m-2} \in G$. Using the same idea, define

$$u_{m-2} = v_{m-1} * h_{n-m+2}(u_{m-1}, \bar{u} \dots, \bar{u})^{-1}$$

with u_{m-1} defined by (12) so as to get

$$\begin{aligned}
v_{m-1} &= u_{m-2} * h_{n-m+2}(u_{m-1}, \bar{u} \dots, \bar{u}) \\
&= u_{m-2} * h_{n-m+2}(u_{m-1}, u_m \dots, u_{k-1}) \\
&= \eta_{m-1}(u_0, \dots, u_k) \\
&= \eta_{m-1}(u_0, \dots, u_{m-3}, \underbrace{v_{m-1} * h_{n-m+2}(u_{m-1}, \bar{u} \dots, \bar{u})^{-1}}_{u_{m-2}}, u_{m-1}, \bar{u} \dots, \bar{u})
\end{aligned}$$

for any choice of $u_0, \dots, u_{m-3} \in G$. Continuing in this way, by induction we obtain elements $u_{m-1}, \dots, u_0 \in G$ such that

$$v_i = \eta_i(u_0, \dots, u_{m-1}, \bar{u} \dots, \bar{u}), \quad i = 1, \dots, m.$$

Therefore, $H_n(u_0, \dots, u_{m-1}, \bar{u} \dots, \bar{u}) = (v_1, \dots, v_m)$ and it follows that H_n is onto G^m .

(b) To show that the SC factorization system consisting of equations (9) and (10) is equivalent to Eq.(1) we show that: (i) each solution $\{x_n\}$ of (1) uniquely generates a solution of the system (9) and (10) and conversely (ii) each solution $\{(t_n, y_n)\}$ of the system (9) and (10) corresponds uniquely to a solution $\{x_n\}$ of (1). To establish (i) let $\{x_n\}$ be the unique solution of (1) corresponding to a given set of initial values $x_0, \dots, x_{-k} \in G$. Define the sequence

$$t_n = x_n * h_n(x_{n-1}, \dots, x_{n-k+m-1}) \quad (13)$$

for $n \geq -m + 1$. Then for each $n \geq 0$ if H_n is defined by (8) it follows from the semiconjugate relation (6) that

$$\begin{aligned}
x_{n+1} &= f_n(x_n, \dots, x_{n-k}) \\
&= \phi_n(x_n * h_n(x_{n-1}, \dots, x_{n-k+m-1}), \dots, \\
&\quad x_{n-m+1} * h_{n-m+1}(x_{n-m}, \dots, x_{n-k})) * [h_{n+1}(x_n, \dots, x_{n-k+m})]^{-1} \\
&= \phi_n(t_n, \dots, t_{n-m+1}) * [h_{n+1}(x_n, \dots, x_{n-k+m})]^{-1}
\end{aligned}$$

Therefore, $\phi_n(t_n, \dots, t_{n-m+1}) = x_{n+1} * h_{n+1}(x_n, \dots, x_{n-k+m}) = t_{n+1}$ so that $\{t_n\}$ is the unique solution of the factor equation (9) with initial values

$$t_{-j} = x_{-j} * h_{-j}(x_{-j-1}, \dots, x_{-j-k+m-1}), \quad j = 0, \dots, m-1.$$

Further, since $x_{n+1} = t_{n+1} * [h_{n+1}(x_n, \dots, x_{n-k+m})]^{-1}$ for $n \geq 0$ by (13), $\{x_n\}$ is the unique solution of the cofactor equation (10) with initial values $y_{-i} = x_{-i}$ for $i = 0, 1, \dots, k-m$ and with the values t_n obtained above.

To establish (ii) let $\{(t_n, y_n)\}$ be a solution of the factor-cofactor system with initial values

$$t_0, \dots, t_{-m+1}, y_{-m}, \dots, y_{-k} \in G.$$

Note that these numbers determine y_{-m+1}, \dots, y_0 through the cofactor equation

$$y_{-j} = t_{-j} * [h_{-j}(y_{-j-1}, \dots, y_{-j-k+m})]^{-1}, \quad j = 0, \dots, m-1. \quad (14)$$

Now for $n \geq 0$ we obtain

$$\begin{aligned} y_{n+1} &= t_{n+1} * [h_{n+1}(y_n, \dots, y_{n-k+m})]^{-1} \\ &= \phi_n(t_n, \dots, t_{n-m+1}) * [h_{n+1}(y_n, \dots, y_{n-k+m})]^{-1} \\ &= \phi_n(y_n * h_n(y_{n-1}, \dots, y_{n-k+m-1}), \dots, \\ &\quad y_{n-m+1} * h_{n-m+1}(y_{n-m}, \dots, y_{n-k})) * h_{n+1}(y_n, \dots, y_{n-k+m})^{-1} \\ &= f_n(y_n, \dots, y_{n-k}) \end{aligned}$$

Thus $\{y_n\}$ is the unique solution of Eq.(1) that is generated by the initial values (14) and y_{-m}, \dots, y_{-k} . This completes the proof of (b).

(c) We show that each coordinate function $\phi_{j,n}$ is the projection into coordinate $j-1$ for $j > 1$. From the definition of H_n in (8) and the semiconjugate relation (6) we infer that

$$\begin{aligned} H_{n+1}(F_n(u_0, \dots, u_k)) &= H_{n+1}(f_n(u_0, \dots, u_k), u_0, \dots, u_{k-1}) \\ &= (f_n(u_0, \dots, u_k) * h_{n+1}(u_0, \dots, u_{k-m}), \\ &\quad u_0 * h_n(u_1, \dots, u_{k-m+1}), \dots, \\ &\quad u_{m-2} * h_{n-m+2}(u_{m-1}, \dots, u_{k-1})). \end{aligned}$$

Matching the corresponding component functions in the above equality for $j \geq 2$ yields

$$\begin{aligned} \phi_{j,n}(u_0 * h_n(u_1, \dots, u_{k+1-m}), \dots, u_{m-1} * h_{n-m+1}(u_m, \dots, u_k)) &= \\ u_{j-2} * h(u_{j-1}, u_j, \dots, u_{j+k-m-1}) \end{aligned}$$

which shows that $\phi_{j,n}$ maps its j -th coordinate to its $(j-1)$ -st. Therefore, for each n and every $(t_1, \dots, t_m) \in H_n(G^{k+1})$ we have

$$\Phi_n(t_1, \dots, t_m) = [\phi_n(t_1, \dots, t_m), t_1, \dots, t_{m-1}]$$

i.e., $\Phi_n|_{H_n(G^{k+1})}$ is of scalar type. Since by Part (a) $H_n(G^{k+1}) = G^m$ for every n , it follows that Φ_n is of scalar type. ■

The pair of equations (9) and (10) in Theorem 2 is uncoupled in the sense that (9) is independent of (10). Such a pair forms a triangular system as defined in [1] and [7]. In the next definition we use convenient and suggestive terminology to describe these equations.

Definition 3 *Eq.(9) is a **factor** of Eq.(1) since it is derived from the semiconjugate factor Φ_n . Eq.(10) that links the factor to the original equation is a **cofactor** of Eq.(1). We refer to the system of equations (9) and (10) as a **semiconjugate (SC) factorization** of Eq.(1). Note that orders m and $k + 1 - m$ of (9) and (10) respectively, add up to the order of (1). We refer to the system of equations (9) and (10) as a **type-($m, k + 1 - m$) order reduction** of Eq.(1).*

3 Invertible-map criterion

In [4] and [5] a useful necessary and sufficient condition is obtained by which to determine whether the difference equation (1) has order-reducing form symmetries (not time-dependent). In this section we show that the same useful idea extends to the time-dependent context. Applications and examples are discussed in the next section.

Consider the following special case of (8) with $m = k$

$$H_n(u_0, u_1, \dots, u_k) = [u_0 * h_n(u_1), u_1 * h_{n-1}(u_2), \dots, u_{k-1} * h_{n-k+1}(u_k)] \quad (15)$$

with $h_n : G \rightarrow G$ being a sequence of surjective self-maps of the underlying group G for $n \geq -k + 1$. If (1) has the form symmetry (15) then it admits a type-($k, 1$) order-reduction and its SC factorization is

$$t_{n+1} = \phi_n(t_n, \dots, t_{n-k+1}), \quad (16)$$

$$x_{n+1} = t_{n+1} * h_{n+1}(x_n)^{-1}. \quad (17)$$

The initial values of the factor equation (16) are

$$t_{-j} = x_{-j} * h_{-j}(x_{-j+1}), \quad j = 0, 1, \dots, k - 1$$

Theorem 4 *(Time-dependent invertible map criterion) Assume that $h_n : G \rightarrow G$ is a sequence of bijections of G for $n \geq -k + 1$. For arbitrary elements $u_0, v_1, \dots, v_k \in G$ and every $n \geq 0$ define $\zeta_{0,n}(u_0) \equiv u_0$ and for $j = 1, \dots, k$,*

$$\zeta_{j,n}(u_0, v_1, \dots, v_j) = h_{n-j+1}^{-1}(\zeta_{j-1,n}(u_0, v_1, \dots, v_{j-1})^{-1} * v_j). \quad (18)$$

with the usual distinction observed between map inversion and group inversion. Then Eq.(1) has the form symmetry $\{H_n\}$ defined by (15) if and only if the quantity

$$f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \dots, \zeta_{k,n}(u_0, v_1, \dots, v_k)) * h_{n+1}(u_0) \quad (19)$$

is independent of u_0 for every $n \geq 0$.

In this case Eq.(1) has a SC factorization whose factor functions in (16) are given by

$$\phi_n(v_1, \dots, v_k) = f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \dots, \zeta_{k,n}(u_0, v_1, \dots, v_k)) * h_{n+1}(u_0). \quad (20)$$

Proof. Assume first that (19) is independent of u_0 for all v_1, \dots, v_k so that the functions

$$\phi_n(v_1, \dots, v_k) = f_n(\zeta_{0,n}, \zeta_{1,n}, \dots, \zeta_{k,n}) * h_{n+1}(u_0) \quad (21)$$

are well defined. Next, if H_n is given by (15) then for all u_0, u_1, \dots, u_k

$$\phi_n(H_n(u_0, u_1, \dots, u_k)) = \phi_n(u_0 * h_n(u_1), u_1 * h_{n-1}(u_2), \dots, u_{k-1} * h_{n-k+1}(u_k)).$$

Now, by (18) for each n and all u_0, u_1

$$\zeta_{1,n}(u_0, u_0 * h_n(u_1)) = h_n^{-1}(u_0^{-1} * u_0 * h_n(u_1)) = u_1.$$

Similarly, for each n and all u_0, u_1, u_2

$$\begin{aligned} \zeta_{2,n}(u_0, u_0 * h_n(u_1), u_1 * h_{n-1}(u_2)) &= h_{n-1}^{-1}(\zeta_{1,n}(u_0, u_0 * h_n(u_1))^{-1} * \\ &\quad u_1 * h_{n-1}(u_2)) \\ &= u_2. \end{aligned}$$

Suppose by way of induction that

$$\zeta_{l,n}(u_0 * h_n(u_1), \dots, u_{l-1} * h_{n-l+1}(u_k)) = u_l$$

for $1 \leq l < j$. Then

$$\zeta_{j,n}(u_0 * h_n(u_1), \dots, u_{j-1} * h_{n-j+1}(u_j)) = h_{n-j+1}^{-1}(u_{j-1}^{-1} * u_{j-1} * h_{n-j+1}(u_j)) = u_j.$$

Thus by (21)

$$\phi_n(H_n(u_0, u_1, \dots, u_k)) = f_n(u_0, \dots, u_k) * h_{n+1}(u_0)$$

Now if F_n and Φ_n are the unfoldings of f_n and ϕ_n respectively, then

$$\begin{aligned} H_{n+1}(F_n(u_0, \dots, u_k)) &= [f_n(u_0, \dots, u_k) * h_{n+1}(u_0), u_0 * h_n(u_1), \\ &\quad \dots, u_{k-2} * h_{n-k+2}(u_{k-1})] \\ &= [\phi_n(H_n(u_0, u_1, \dots, u_k)), u_0 * h_n(u_1), \\ &\quad \dots, u_{k-2} * h_{n-k+2}(u_{k-1})] \\ &= \Phi_n(H_n(u_0, \dots, u_k)) \end{aligned}$$

and it follows that $\{H_n\}$ is a semiconjugate form symmetry for Eq.(1). The existence of a SC factorization with factor functions defined by (20) now follows from Lemma 2.

Conversely, if $\{H_n\}$ as given by (15) is a time-dependent form symmetry of Eq.(1) then the semiconjugate relation implies that for arbitrary u_0, \dots, u_k in G there are functions ϕ_n such that

$$f_n(u_0, \dots, u_k) * h_{n+1}(u_0) = \phi_n(u_0 * h_n(u_1), \dots, u_{k-1} * h_{n-k+1}(u_k)). \quad (22)$$

For every u_0, v_1, \dots, v_k in G and with functions $\zeta_{j,n}$ as defined above, note that

$$\zeta_{j-1,n}(u_0, v_1, \dots, v_{j-1}) * h_{n-j+1}(\zeta_{j,n}(u_0, v_1, \dots, v_j)) = v_j, \quad j = 1, 2, \dots, k.$$

Therefore, abbreviating $\zeta_{j,n}(u_0, v_1, \dots, v_j)$ by $\zeta_{j,n}$ we have

$$\begin{aligned} f_n(\zeta_{0,n}, \zeta_{1,n}, \dots, \zeta_{k,n}) * h_{n+1}(u_0) &= \phi_n(\zeta_{0,n} * h_n(\zeta_{1,n}), \zeta_{1,n} * h_{n-1}(\zeta_{2,n}), \\ &\quad \dots, \zeta_{k-1,n} * h_{n-k+1}(\zeta_{k,n})) \\ &= \phi_n(v_1, \dots, v_k) \end{aligned}$$

which is independent of u_0 . ■

Recall that an algebraic field $\mathcal{F} = (\mathcal{F}, +, \cdot)$ is, in particular, a commutative group with respect to addition. Further, its set of nonzero elements $\mathcal{F} \setminus \{0\}$ is a commutative group under multiplication. A simple yet important type of form symmetry may be defined on a field.

Definition 5 Let \mathcal{F} be a non-trivial field and $\{\alpha_n\}$ a sequence of elements of \mathcal{F} such that $\alpha_n \in \mathcal{F} \setminus \{0\}$ for all $n \geq -k+1$. A (time-dependent) **linear form symmetry** is defined as the following special case of (15) with $h_n(u) = -\alpha_{n-1}u$

$$[u_0 - \alpha_{n-1}u_1, u_1 - \alpha_{n-2}u_2, \dots, u_{k-1} - \alpha_{n-k}u_k]. \quad (23)$$

The sequence $\{\alpha_n\}$ of nonzero elements in \mathcal{F} may be called the **eigensequence** of the linear form symmetry. If Eq.(1) has a linear form symmetry then $\{\alpha_n\}$ is an eigensequence of (1).

The use of the term “eigen” which is borrowed from the theory of linear equations is apt here for two reasons. First, the sequence $\{\alpha_n\}$ characterizes the linear form symmetry (23) completely and secondly, we find below that linear difference equations indeed have linear form symmetries.

The existence of a linear form symmetry implies a type- $(k, 1)$ order reduction for Eq.(1) and a SC factorization where the cofactor equation (17) is determined more specifically as

$$x_{n+1} = t_{n+1} + \alpha_n x_n. \quad (24)$$

The following necessary and sufficient condition for the existence of a time-dependent linear form symmetry is an application of Theorem 4. We drop further mention of “type- $(k, 1)$ ” as we do not discuss any other order reduction types in the remainder of this paper.

Corollary 6 Equation (1) has a time-dependent linear form symmetry of type (23) with an eigensequence $\{\alpha_n\}$ in a non-trivial field \mathcal{F} if and only if the quantity

$$f_n(u_0, \zeta_{1,n}(u_0, v_1), \dots, \zeta_{k,n}(u_0, v_1, \dots, v_k)) - \alpha_n u_0 \quad (25)$$

is independent of u_0 for all $n \geq 0$ with the functions $\zeta_{j,n}$ for $j = 1, \dots, k$ given by

$$\begin{aligned} \zeta_{j,n}(u_0, v_1, \dots, v_j) &= \frac{\zeta_{j-1,n}(u_0, v_1, \dots, v_{j-1}) - v_j}{\alpha_{n-j}} \\ &= \frac{1}{\prod_{i=1}^j \alpha_{n-i}} \left(u_0 - \sum_{i=1}^j v_i \prod_{p=1}^i \alpha_{n-p} \right). \end{aligned}$$

Proof. The conclusions follow immediately from Theorem 4 using $h_n(u) = -\alpha_{n-1}u$. The last equality above is established from the equality preceding it by routine calculation. ■

Remark 7 If Eq.(1) has a linear form symmetry then by Corollary 6 an eigensequence of (1) can be defined equivalently as a sequence $\{\alpha_n\}$ in $\mathcal{F} \setminus \{0\}$ for which the quantity in (25) is independent of u_0 for every $n \geq 0$.

We close this section with an example of a nonlinear equation that has a linear form symmetry. For additional results and examples, we refer to [2].

Example 8 Consider the following third-order nonlinear difference equation

$$x_{n+1} = (-1)^{n+1}x_n - 2x_{n-1} + g_n(x_n + x_{n-2}) \quad (26)$$

where $g_n : \mathbb{R} \rightarrow \mathbb{R}$ is a given function for each $n \geq -2$. By Corollary 6 a linear form symmetry for (26) exists if and only if the quantity

$$(-1)^{n+1}u_0 - 2\zeta_{1,n} + g_n(u_0 + \zeta_{2,n}) - \alpha_n u_0 \quad (27)$$

is independent of u_0 for all n . Substituting

$$\zeta_{1,n} = \frac{u_0 - v_1}{\alpha_{n-1}}, \quad \zeta_{2,n} = \frac{\zeta_{1,n} - v_2}{\alpha_{n-2}} = \frac{u_0 - v_1 - \alpha_{n-1}v_2}{\alpha_{n-1}\alpha_{n-2}}.$$

in (27) and rearranging terms gives

$$\begin{aligned} &\left((-1)^{n+1} - \frac{2}{\alpha_{n-1}} - \alpha_n \right) u_0 + \frac{2}{\alpha_{n-1}} v_1 + \\ &g_n \left(\left(1 + \frac{1}{\alpha_{n-1}\alpha_{n-2}} \right) u_0 - \frac{1}{\alpha_{n-1}\alpha_{n-2}} v_1 - \frac{1}{\alpha_{n-2}} v_2 \right) \end{aligned}$$

which is independent of u_0 for all n if the coefficients of the u_0 terms are zeros; i.e., for all n , the numbers α_n satisfy both of the following equations

$$\alpha_n = (-1)^{n+1} - \frac{2}{\alpha_{n-1}} \quad (28)$$

$$\alpha_{n-1} = -\frac{1}{\alpha_{n-2}}. \quad (29)$$

Every solution of Eq.(29) is a sequence of period 2

$$\left\{ q, -\frac{1}{q}, q, -\frac{1}{q}, \dots \right\} \quad (30)$$

where $q = \alpha_{-2} \in \mathbb{R}$. Now (29) yields $\alpha_{-1} = -1/q$, which we substitute as an initial value in Eq.(28) to get

$$\alpha_0 = -1 + 2q.$$

Now to make the period-two sequence in (30) also a solution of (28), we require the above value of α_0 to be equal to q ; thus

$$\alpha_0 = q \Rightarrow 2q - 1 = q \Rightarrow q = 1.$$

We check that if $\alpha_0 = q = 1$ in (28) then

$$\alpha_1 = 1 - \frac{2}{q} = -1 = -\frac{1}{q}, \quad \alpha_2 = -1 - \frac{2}{-1} = 1 = q, \quad \text{etc}$$

so that both of the equations (28) and (29) generate the same sequence $\{\alpha_n\}$ where $\alpha_n = (-1)^n$ for $n \geq -2$. It follows that $\{(-1)^n\}$ is an eigensequence for (26).

4 Factorization of linear equations

We expect that linear difference equations are among difference equations that have the linear form symmetry and this is indeed the case. The following application of Corollary 6 and Theorem 4 gives the semiconjugate factorization for non-autonomous and non-homogeneous linear difference equations.

Corollary 9 (The general linear equation) Let $\{a_{i,n}\}$, $i = 1, \dots, k$ and $\{b_n\}$ be given sequences in a non-trivial field \mathcal{F} such that $a_{k,n} \neq 0$ for all $n \geq 0$. The non-homogeneous linear equation of order $k + 1$

$$x_{n+1} = a_{0,n}x_n + a_{1,n}x_{n-1} + \dots + a_{k,n}x_{n-k} + b_n \quad (31)$$

has a linear form symmetry with eigensequence $\{\alpha_n\}$ for every solution $\{x_n\}$ in \mathcal{F} of the following Riccati equation of order k

$$\alpha_n = a_{0,n} + \frac{a_{1,n}}{\alpha_{n-1}} + \frac{a_{2,n}}{\alpha_{n-1}\alpha_{n-2}} + \dots + \frac{a_{k,n}}{\alpha_{n-1} \cdots \alpha_{n-k}} \quad (32)$$

The corresponding SC factorization of (31) is

$$t_{n+1} = b_n - \sum_{i=1}^k \sum_{j=i}^k \frac{a_{j,n}}{\alpha_{n-i} \cdots \alpha_{n-j}} t_{n-i+1} \quad (33)$$

$$x_{n+1} = \alpha_n x_n + t_{n+1} \quad (34)$$

Proof. By Corollary 6 it is only necessary to determine a sequence $\{\alpha_n\}$ of nonzero elements of \mathcal{F} such that for each n the quantity (25) is independent of u_0 for the following function

$$f_n(u_0, \dots, u_k) = a_{1,n}u_0 + a_{2,n}u_1 + \cdots + a_{k,n}u_k + b_n.$$

For arbitrary $u_0, v_1, \dots, v_k \in \mathcal{F}$ and $j = 0, 1, \dots, k$ define $\zeta_{j,n}(u_0, v_1, \dots, v_j)$ as in Corollary 6. Then the expression (25) is

$$\begin{aligned} & -\alpha_n u_0 + b_n + a_{1,n}u_0 + a_{2,n}\zeta_{1,n}(u_0, v_1) + \cdots + a_{k,n}\zeta_{k,n}(u_0, v_1, \dots, v_k) = \\ & b_n + \left[\sum_{j=1}^k \frac{a_{j,n}}{\prod_{i=1}^j \alpha_{n-i}} - \alpha_n \right] u_0 - \sum_{j=1}^k a_{j,n} \sum_{i=1}^j \frac{v_i}{\prod_{p=i}^j \alpha_{n-p}} \end{aligned}$$

The above quantity is independent of u_0 if and only if the coefficient of u_0 is zero for all n ; i.e., if $\{\alpha_n\}$ is a solution of the Riccati difference equation

$$\alpha_n = \sum_{j=1}^k \frac{a_{j,n}}{\prod_{i=1}^j \alpha_{n-i}}$$

which is Eq.(32). It follows that Eq.(31) has a linear form symmetry of type (23) with eigensequence $\{\alpha_n\}$ for each solution $\{\alpha_n\}$ of the Riccati equation. For the corresponding SC factorization of (31), the cofactor equation is simply (24) while the factor equation is obtained using the above calculations and Eq.(20) of Theorem 4 as follows

$$\begin{aligned} t_{n+1} &= b_n - \sum_{j=1}^k a_{j,n} \sum_{i=1}^j \frac{t_{n-i+1}}{\prod_{p=i}^j \alpha_{n-p}} \\ &= b_n - \sum_{i=1}^k \sum_{j=i}^k \frac{a_{j,n}}{\alpha_{n-i} \cdots \alpha_{n-j}} t_{n-i+1}. \end{aligned}$$

This completes the proof. ■

Corollary 9 states that *any* solution of the Riccati equation (32) gives a form symmetry and a SC factorization of (31) as specified above. The next example illustrates Corollary 9.

Example 10 Consider the second-order difference equation

$$x_{n+1} = (-1)^{n+1}x_n + x_{n-1} + b_n \quad (35)$$

where b_n, x_0, x_{-1} are in a field \mathcal{F} which we may take to be any one of the familiar fields \mathbb{Q}, \mathbb{R} or \mathbb{C} . The associated Riccati equation of (35) is

$$\alpha_n = (-1)^{n+1} + \frac{1}{\alpha_{n-1}}. \quad (36)$$

Straightforward calculation shows that if $\alpha_0 \neq 0, -1$ then

$$\begin{aligned} \alpha_1 &= \frac{\alpha_0 + 1}{\alpha_0}, \quad \alpha_2 = -\frac{1}{\alpha_0 + 1}, \quad \alpha_3 = -\alpha_0, \\ \alpha_4 &= -\frac{\alpha_0 + 1}{\alpha_0}, \quad \alpha_5 = \frac{1}{\alpha_0 + 1}, \quad \alpha_6 = \alpha_0. \end{aligned}$$

It follows that all solutions of the Riccati equation (36) with initial value outside the singularity set $\{0, -1\}$ are eigensequences in \mathcal{F} of period 6:

$$\left\{ \alpha_0, \frac{\alpha_0 + 1}{\alpha_0}, -\frac{1}{\alpha_0 + 1}, -\alpha_0, -\frac{\alpha_0 + 1}{\alpha_0}, \frac{1}{\alpha_0 + 1}, \alpha_0, \dots \right\}$$

The SC factorization of the linear equation (35) is now obtained by Corollary 9 as

$$\begin{aligned} t_{n+1} &= -\frac{1}{\alpha_{n-1}}t_n + b_n, \\ x_{n+1} &= \alpha_n x_n + t_{n+1}. \end{aligned}$$

The next result is concerned with the case of constant coefficients. The straightforward proof is omitted.

Corollary 11 Let $\{b_n\}$ be a given sequence in a non-trivial field \mathcal{F} and let $\{a_i\}$, $i = 1, \dots, k$ be constants in \mathcal{F} such that $a_k \neq 0$.

(a) The non-homogeneous linear equation of order $k + 1$

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \dots + a_k x_{n-k} + b_n \quad (37)$$

has a linear form symmetry with eigensequence $\{\alpha_n\}$ for every solution $\{\alpha_n\}$ in \mathcal{F} of the following autonomous Riccati equation of order k

$$\alpha_n = a_0 + \frac{a_1}{\alpha_{n-1}} + \frac{a_2}{\alpha_{n-1}\alpha_{n-2}} + \dots + \frac{a_k}{\alpha_{n-1}\dots\alpha_{n-k}}. \quad (38)$$

(b) Every fixed point of (38) in \mathcal{F} is a nonzero root of the characteristic polynomial of (37), i.e.,

$$\lambda^{k+1} - a_0 \lambda^k - a_1 \lambda^{k-1} - \dots - a_{k-1} \lambda - a_k \quad (39)$$

and thus, an eigenvalue of the homogeneous part of (37) in \mathcal{F} . As constant solutions of (38) such eigenvalues are constant eigensequences of (37).

Example 12 Consider the autonomous second-order linear difference equation

$$x_{n+1} = x_n + x_{n-1} \quad (40)$$

Eq.(40) has two real eigenvalues

$$\alpha_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

as roots of the characteristic polynomial $\lambda^2 - \lambda - 1$ or equivalently, as fixed points of the Riccati equation

$$\alpha_n = 1 + \frac{1}{\alpha_{n-1}} \quad (41)$$

in the field $\mathcal{F} = \mathbb{R}$. Thus each of α_+ and α_- is a constant eigensequence of (40) in \mathbb{R} and the following SC factorization is obtained in \mathbb{R} :

$$\begin{aligned} t_{n+1} &= -\frac{1}{\alpha_+} t_n = \alpha_- t_n, \\ x_{n+1} &= \alpha_+ x_n + t_{n+1}. \end{aligned}$$

Note that the SC factorization above has constant coefficients also. We note further that since α_{\pm} are irrational the above SC factorization is not valid if $\mathcal{F} = \mathbb{Q}$ the field of rational numbers. In fact, since the characteristic polynomial has no rational roots, it follows that there are no constant eigensequences for (40) in \mathbb{Q} . However, Riccati equation (41) is a rational equation and thus with a rational initial value α_0 the corresponding solution of (41) is a solution (non-constant) in \mathbb{Q} . For instance, if $\alpha_0 = 1$ then the corresponding solution of (41) is $\alpha_n = \varphi_{n+1}/\varphi_n$ where $\{\varphi_n\}$ is the Fibonacci sequence $1, 1, 2, 3, 5, 8, \dots$. This rational eigensequence yields the following SC factorization of (40) that is valid in \mathbb{Q} :

$$\begin{aligned} t_{n+1} &= -\frac{\varphi_{n-1}}{\varphi_n} t_n, \\ x_{n+1} &= \frac{\varphi_{n+1}}{\varphi_n} x_n + t_{n+1}. \end{aligned}$$

We note that $\lim_{n \rightarrow \infty} \varphi_{n+1}/\varphi_n = \alpha_+$ in the above factorization; in this way the factorization over rationals is related to the earlier factorization over the reals. In a similar fashion, the equation

$$x_{n+1} = x_n - x_{n-1} \quad (42)$$

has two complex eigenvalues

$$\alpha_{\pm} = \frac{1 \pm i\sqrt{3}}{2}$$

that are roots of $\lambda^2 - \lambda + 1$. Thus, (42) has no constant eigensequences in \mathbb{R} but it does have non-constant real eigensequences since the Riccati equation

$$\alpha_n = 1 - \frac{1}{\alpha_{n-1}}$$

with the initial value $\alpha_0 = 2$ has a solution

$$\left\{ 2, \frac{1}{2}, -1, 2, \frac{1}{2}, -1, \dots \right\}$$

of period three in \mathbb{R} with a corresponding real SC factorization

$$t_{n+1} = -\frac{1}{\alpha_{n-1}}t_n, \quad x_{n+1} = \alpha_n x_n + t_{n+1}.$$

In contrast to the factorization of Eq.(40) there is no simple relationship between the factorization of (42) over the real numbers and its factorization with constant eigensequences over the complex numbers.

Remark 13 Is it possible that a linear difference equation has no eigensequences, constant or otherwise in a given field \mathcal{F} because the associated Riccati equation has no solutions at all in \mathcal{F} ?

We know the answer to this question in some cases. If we have a linear equation (homogeneous or not) with constant coefficients in an algebraically closed field \mathcal{F} (e.g., the field \mathbb{C} of complex numbers) then \mathcal{F} always contains constant eigensequences, namely, the roots of the characteristic polynomial (39). On the other hand, for the finite field $\mathbb{Z}_3 = \{0, 1, 2\}$ with addition and multiplication defined modulo 3, the linear equation (40) has no eigensequences. This can be shown by testing each of the two possible nonzero initial values 1, 2 in the Riccati equation (41) to verify that both lead to the singularity at 0:

$$\begin{aligned} \alpha_0 = 2 &\Rightarrow \alpha_1 = 1 + \frac{1}{2} = 1 + 2 = 0, \\ \alpha_0 = 1 &\Rightarrow \alpha_1 = 1 + 1 = 2 \Rightarrow \alpha_2 = 0. \end{aligned}$$

The answer to the question of existence of eigensequences in the general case is not known at this time; in fact, it is not known if a linear equation with real coefficients exists that has no real eigensequences. For “large” fields such as \mathbb{R} or \mathbb{C} it seems likely that the general linear equation (31) has an eigensequence in the field.

The occurrence of Riccati difference equation in Corollary 9 may seem less surprising if we recall some basic facts from [5]. In particular, the homogeneous part of (31) is a homogeneous equation of degree one relative to the multiplicative group $\mathcal{F} \setminus \{0\}$. Therefore, it has an inversion form symmetry and the factor equation of its SC factorization is none other than the Riccati equation (32). Using this fact it is possible to restate Corollary 9 without explicit reference to the Riccati equation as follows.

Corollary 14 Assume that the homogeneous part of Eq.(31) has a solution $\{y_n\}$ in the field \mathcal{F} such that $y_n \neq 0$ for all n . Then $\{y_{n+1}/y_n\}$ is an eigensequence of (31) whose SC factorization is given by the pair of equations (33) and (34).

Proof. It is given that $\{y_n\}$ satisfies the homogeneous part of (31), i.e.,

$$y_{n+1} = a_{0,n}y_n + a_{1,n}y_{n-1} + a_{2,n}y_{n-2} + \cdots + a_{k,n}y_{n-k}.$$

Since $y_n \neq 0$ for all n , we may divide the above equation by y_n to obtain

$$\begin{aligned} \frac{y_{n+1}}{y_n} &= a_{0,n} + a_{1,n} \frac{y_{n-1}}{y_n} + a_{2,n} \frac{y_{n-2}}{y_n} + \cdots + a_{k,n} \frac{y_{n-k}}{y_n} \\ &= a_{0,n} + a_{1,n} \frac{y_{n-1}}{y_n} + a_{2,n} \frac{y_{n-2}}{y_{n-1}} \frac{y_{n-1}}{y_n} + \cdots + a_{k,n} \frac{y_{n-k}}{y_{n-k+1}} \cdots \frac{y_{n-1}}{y_n}. \end{aligned}$$

Now defining $\alpha_n = y_{n+1}/y_n$ for all n and substituting these terms in the last equation above yields the Riccati equation (32). Thus $\{y_{n+1}/y_n\}$ is an eigensequence of (31) in $\mathcal{F} \setminus \{0\}$, as claimed. The SC factorization is obtained as in the proof of Corollary 9. ■

Corollary 15 In Eq.(31) let $\{a_{i,n}\}$, $i = 1, \dots, k$ and $\{b_n\}$ be sequences of real numbers with $a_{i,n} \geq 0$ for all i, n and $a_{k,n} > 0$ for all n . Then (31) has an eigensequence $\{y_{n+1}/y_n\}$ and a SC factorization in \mathbb{R} given by the pair of equations (33) and (34).

Proof. If we choose $y_{-j} = 1$ for $j = 0, \dots, k$ then the corresponding solution $\{y_n\}$ of the homogeneous part of (31) is a sequence of positive real numbers. Now an application of Corollary 14 completes the proof. ■

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